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A New Global Bifurcation Theorem and Its Applications in Non-Cooperative Models of Nuclear Reactors

CHEN Ruipeng

School of Mathematics and Information Sciences, North Minzu University, Yinchuan 750021, Ningxia, China

Abstract: In this work, we establish a version of the global bifurcation theorem for nonlinear Sturm-Liouville problem

$$\begin{cases} -(pu')'+qu=\gamma f(t,u)+g(t,u,u',\gamma), & t\in(0,1),\\ b_0u(0)+c_0u'(0)=0,\ b_1u(1)+c_1u'(1)=0, \end{cases}$$

where f is not Fréchet differentiable at 0. We show that the bifurcation occurs from some intervals $I_k(k\in\mathbb{N})$; the connected component D_k^{γ} containing $I_k\times\{0\}$ is unbounded and lies in $(\mathbb{R}\times S_k^{\gamma})\cup(I_k\times\{0\})$. By applying the above theorem, we prove that the non-cooperative model $\begin{cases} -u''=uv-\lambda u+f(t,u), & t\in(0,1),\\ -v''=\mu u, & t\in(0,1), \end{cases}$ admits at least one positive solution for any $\mu\in(0,\infty)$.

Key words: non-cooperative model; nuclear reactor; bifurcation; positive solution; existence

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0 Introduction

In the past few decades, there has been considerable interest in studying the time and space-dependent mathematical model of nuclear reactors in a closed container[1],

$$\begin{cases} \dfrac{\partial u}{\partial t}-D\Delta u=u(av-b), & x\in\Omega,\ t>0,\\ \dfrac{\partial v}{\partial t}=cu, & x\in\Omega,\ t>0, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^n$ is a smooth bounded domain and indicates the closed container, u and v represent the density of the neutron flux and temperature of the nuclear reactors, respectively. a, b, c and D are constants with $b\in[0,\infty)$ and $c\in(0,\infty)$. $a>0$ means the positive feedback, $a<0$ means the negative one, and D indicates the diffusion coefficient.

If we consider the nuclear reactors with no neutron flux on the boundary of the container and the constant temperature on it, then the model to describe the neutron flux and temperature of the nuclear reactors is

$$\begin{cases} \dfrac{\partial u}{\partial t}-\Delta u=u(av-b), & x\in\Omega,\ t>0,\\ \dfrac{\partial v}{\partial t}-\Delta v=cu, & x\in\Omega,\ t>0,\\ u=v=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x)\geqslant 0,\ v(x,0)=v_0(x)\geqslant 0, & x\in\bar{\Omega}, \end{cases} \tag{2}$$

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Biography: CHEN Ruipeng, male, Ph. D., Lecturer, research direction: nonlinear differential equations and systems. E-mail: ruipengchen@126.com

where u_0 and v_0 are continuous nonnegative functions on $\bar{\Omega}$. Existence of positive stationary solutions and the long term dynamics of (2) and its general forms have been studied by many authors[1-16]. Owing to the close relationship between the dynamics of (2) and its associated steady states, several authors have also paid their attentions to the stationary version and one-dimensional analogue of (2)[8,10-13,16].

By applying the fixed point theory, Wang and An[13] studied the existence of positive solutions of the non-cooperative system

$$\begin{cases} -u''=uv-\lambda u+f(t,u), & t\in(0,1),\\ -v''=\mu u, & t\in(0,1),\\ u(0)=u(1)=0,\\ v(0)=v(1)=0, \end{cases} \tag{3}$$

where $\lambda>-\pi^2$ is a constant and $\mu>0$ is a parameter. Eq. (3) is related to the stationary version of the reaction-diffusion system (2). They proved (3) has at least one positive solution for $\mu\in(0,C_0]$, where C_0 is a constant. Then Chen and Ma[16] investigated the existence of positive solutions of (3) with $f(t,u)=g(t)f(u)$. By using bifurcation techniques, they showed (3) admits at least one positive solution for any $\mu\in(0,\infty)$, which extends the results of Wang and An[13] to some extent. However, above results were established under the assumption that f is nonnegative and $\lim\limits_{s\to 0+}\dfrac{f(s)}{s}$ exists.

A natural question is whether (3) admits a positive solution for any $\mu\in(0,\infty)$ or f is not necessarily linearizable at 0. In addition, the fourth open problem listed by Chen[8] suggested that the existence results for (3) should be established with more general nonlinearities. Motivated by the above considerations, we will give a positive answer to the above question. More precisely, we shall investigate the existence of positive solutions of (3) under the following assumptions

(H1) $\lambda\in(-\pi^2,\infty)$ is a constant.

(H2) $f\in C([0,1]\times\mathbb{R},\mathbb{R})$ with $sf(t,s)>0$ for $s\neq 0$ and $t\in[0,1]$, and there exist functions $f_0(\cdot), f^0(\cdot)\in C([0,1],(0,\infty))$ such that

$$\liminf_{|s|\to 0}\frac{f(t,s)}{s}=f_0(t), \text{ uniformly for } t\in[0,1],$$

$$\limsup_{|s|\to 0}\frac{f(t,s)}{s}=f^0(t), \text{ uniformly for } t\in[0,1].$$

(H3) There exists a function $c(\cdot)\in C([0,1],(0,\infty))$ such that

$$f(t,u)\geqslant c(t)u,\ (t,u)\in[0,1]\times[0,\infty).$$

The rest of this paper is organized as follows. In Section 1, we state and prove the global bifurcation theorems for nonlinear Sturm-Liouville problems

$$\begin{cases} -(pu')'+qu=\gamma f(t,u)+g(t,u,u',\gamma), & t\in(0,1),\\ b_0u(0)+c_0u'(0)=0,\ b_1u(1)+c_1u'(1)=0, \end{cases} \tag{4}$$

and

$$\begin{cases} -(pu')'+qu=\gamma h(t)f(u)+g(t,u,u',\gamma), & t\in(0,1),\\ b_0u(0)+c_0u'(0)=0,\ b_1u(1)+c_1u'(1)=0, \end{cases} \tag{5}$$

where f is not necessarily linearizable at 0, and $h\in C([0,1],[0,\infty))$ is a nonnegative weight function. Section 2 is devoted to establishing some existence results of positive solutions for (3), and also for (3) with $f(t,u)=h(t)f(u)$, via the above bifurcation theorems. In Section 3, we give some related results.

1 Global Bifurcation Theorem

Let (b.c.) denote the boundary conditions

$$b_0u(0)+c_0u'(0)=0,\ b_1u(1)+c_1u'(1)=0,$$

where b_0, b_1, c_0 and c_1 are real numbers with $|b_i|+|c_i|\neq 0$, $i=0,1$. We denote by L the Sturm-Liouville differential operator defined as $Lu=-(pu')'+qu$, where $p\in C^1[0,1]$ with $p>0$ on $[0,1]$, $q\in C[0,1]$ and

$$D(L)=\{u\in C^2[0,1]: u \text{ satisfies (b.c.)}\}.$$

Berestycki[17] established a global bifurcation theorem for the Sturm-Liouville problems

$$\begin{cases} Lu=\gamma a(t)u+F(t,u,u',\gamma), & t\in(0,1),\\ \text{(b.c.)}, \end{cases} \tag{6}$$

where $a\in C([0,1],(0,\infty))$, $F=f_1+g_1$, f_1 and g_1 are continuous functions on $[0,1]\times\mathbb{R}^3$ satisfying

$$|f_1(t,u,s,\gamma)|\leqslant M|u|,\ (t,u,s,\gamma)\in[0,1]\times[-1,1]^2\times\mathbb{R}, \tag{7}$$

and

$$g_1(t,u,s,\gamma)=o(|u|+|s|),\ \text{near}\ (u,s)=(0,0), \tag{8}$$

uniformly for $(t,\gamma)\in[0,1]\times\Gamma$, for every bounded interval Γ. It is not difficult to see this problem need not be linearizable in any neighborhood of the origin. Above results have been extended by Rynne[18] under the assumption

$$|f_1(t,u,s,\gamma)|\leqslant M_0|u|+M_1|s|,\ (t,u,s,\gamma)\in[0,1]\times\mathbb{R}^3, \tag{9}$$

where M_0 and M_1 are some positive constants. However, it is clear the nonlinear term $\gamma f(t,u)$ in (4) does not meet the condition (7) or (9) since there is no $M>0$ such that

$$|\gamma f(t,u)|\leqslant M|u| \text{ for all } (t,u,\gamma)\in[0,1]\times\mathbb{R}^2. \tag{10}$$

Therefore, this section is devoted to establishing a version of the global bifurcation theorem for nonlinear Sturm-Liouville problem (4).

Let a be a positive continuous function on $[0,1]$. It

is a classical result[19] that the linear Sturm-Liouville problem

$$\begin{cases} Lu=\gamma a(t)u,\ t\in(0,1), \\ \text{(b.c.)}, \end{cases} \tag{11}$$

possesses infinitely many simple eigenvalues $\{\gamma_k(a)\}_{k=1}^{\infty}$ with

$$0<\gamma_1(a)<\cdots<\gamma_k(a)<\cdots$$

and $\lim\limits_{k\to\infty}\gamma_k(a)=+\infty$. The zeros in [0, 1] of any eigenfunction ϕ_k corresponding to $\gamma_k(a)$ are nodes (i.e., points ξ, where $\phi_k(\xi)=0, \phi_k'(\xi)\neq 0$), and ϕ_k has exactly $k-1$ zeros in (0, 1).

Let $Y=C[0,1]$ with the norm $\|u\|_\infty=\max\limits_{t\in[0,1]}|u(t)|$ and E the Banach space

$$E=\{u\in C^1[0,1]\text{: } u \text{ satisfies (b.c.)}\}$$

equipped with the norm $\|u\|_1=\max\{\|u\|_\infty,\ \|u'\|_\infty\}$. S_k^+ denotes the set of functions $u\in E$ which have exactly $k-1$ zeros in (0, 1), all of whose zeros in [0, 1] are nodal, and which are positive in a deleted neighborhood of 0. The sets S_k^+, $S_k^-=-S_k^+$, and $S_k=S_k^+\cup S_k^-$ are open sets in E. In the following, we denote by ϕ_k^+ the unique eigenfunction of (11) associated to γ_k such that $\phi_k^+\in S_k^+$ and $\|\phi_k^+\|_1=1$; we also let $\phi_k^-=-\phi_k^+$. We denote by φ the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions of (4) and φ_k^ν the closure in $\mathbb{R}\times E$ of the set of all solutions (γ,u) of (4) with $u\in S_k^\nu$, where ν denotes + or −.

Define

$$\alpha(t)=\frac{f^0(t)+f_0(t)}{2},\ \beta(t)=\frac{f^0(t)-f_0(t)}{2}, \tag{12}$$

where $f^0(\cdot)$ and $f_0(\cdot)$ are functions given by (H2). It follows from (H2) that $\alpha\in C([0,1],\ (0,\infty))$ and $\beta\in C([0,1],\ [0,\infty))$.

We suppose throughout the section that

(H4) $\alpha_0>\beta^0$, where $\alpha_0=\min\limits_{t\in[0,1]}\alpha(t)>0$ and $\beta^0=\max\limits_{t\in[0,1]}\beta(t)>0$.

(H5) $g(t,u,s,\gamma)=o(|u|+|s|)$, near $(u,s)=(0,0)$, uniformly in $t\in[0,1]$ and in $\gamma\in\Gamma$, for every bounded interval Γ.

For each $k\in\mathbb{N}$, set

$$I_k=\left[\frac{\gamma_k(\alpha)\alpha_0}{\alpha_0+\beta^0},\ \frac{\gamma_k(\alpha)\alpha_0}{\alpha_0-\beta^0}\right], \tag{13}$$

where $\gamma_k(\alpha)$ is the k-th eigenvalue of the linear problem

$$\begin{cases} Lu=\gamma\alpha(t)u,\ t\in(0,1), \\ \text{(b.c.)}. \end{cases}$$

Theorem 1 Let (H2), (H4) and (H5) hold. Then for each $k\in\mathbb{N}$ and each $\nu\in\{+,-\}$, the connected component D_k^ν of $\varphi_k^\nu\cup(I_k\times\{0\})$ containing $I_k\times\{0\}$ is unbounded, and lies in $(\mathbb{R}\times S_k^\nu)\cup(I_k\times\{0\})$.

Proof It suffices to indicate why Berestycki's proof[17] is still valid here. Let

$$h(t,s)=f(t,s)-\alpha(t)s,\ (t,s)\in[0,1]\times\mathbb{R}. \tag{14}$$

Then

$$\liminf_{|s|\to 0}\frac{h(t,s)}{s}=-\beta(t),\ \limsup_{|s|\to 0}\frac{h(t,s)}{s}=\beta(t), \tag{15}$$

uniformly for $t\in[0,1]$, and (4) can be rewritten as

$$\begin{cases} Lu=\gamma\alpha(t)u+\gamma h(t,u)+g(t,u,u',\gamma),\ t\in(0,1), \\ \text{(b.c.)}. \end{cases} \tag{16}$$

Consider the auxiliary problem

$$\begin{cases} Lu=\gamma\alpha(t)u+\gamma h(t,|u|^\varepsilon u)+g(t,u,u',\gamma),\ t\in(0,1), \\ \text{(b.c.)}, \end{cases} \tag{17}$$

where $\varepsilon\in[0,1]$ is a parameter. For the auxiliary problem (17), we claim that the following results hold.

Let ε_n converge to 0 and satisfy $0\leqslant\varepsilon_n\leqslant 1$. If there is a sequence $(\gamma_n,u_n)\in\mathbb{R}\times S_k^\nu$ such that (γ_n,u_n) is a solution of (17) corresponding to $\varepsilon=\varepsilon_n$, and

$$(\gamma_n,u_n)\to(\gamma,0)\text{ in }\mathbb{R}\times E, \tag{18}$$

Then $\gamma\in I_k$.

In fact, it is clear in (18) $\gamma_n\to\gamma$ as $n\to\infty$, so $\{\gamma_n\}$ is bounded, and thus the proof of the above results is similar to that of Lemma 1 in Ref. [17] with obvious modifications, even though there is no $M>0$ such that (10) holds. Finally, by above results we can easily complete the proof.

Remark 1 Note that if $f^0(t)\equiv f_0(t)=f_0\in(0,\infty)$ and $g(t,u,u',\gamma)\equiv 0$, then (4) reduces to the corresponding nonlinear eigenvalue problems[20-24]. Meanwhile, we have $\alpha(t)\equiv\alpha_0=f_0$, $\beta(t)\equiv\beta^0=0$ and (H4) always holds. Moreover, $I_k\times\{0\}$ will reduce to the single point $(\gamma_k(f_0),\ 0)=\left(\frac{\mu_k}{f_0},\ 0\right)$, from which bifurcation occurs. Here μ_k is the k-th eigenvalue of the linear Sturm-Liouville problem

$$\begin{cases} Lu=\mu u,\ t\in(0,1), \\ \text{(b.c.)}. \end{cases} \tag{19}$$

Remark 2 If $f^0(t)\equiv\bar{f}_0\in(0,\infty)$, $f_0(t)\equiv\underline{f}_0\in(0,\infty)$, then

$$\alpha(t)\equiv\alpha_0=\frac{\bar{f}_0+\underline{f}_0}{2},\ \beta(t)\equiv\beta^0=\frac{\bar{f}_0-\underline{f}_0}{2},$$

and (H4) is still satisfied. Clearly, I_k reduces to $\left[\frac{\mu_k}{\bar{f}_0},\ \frac{\mu_k}{\underline{f}_0}\right]$, from which bifurcation occurs.

For the nonlinear Sturm-Liouville problem (5), we have the following theorem.

Theorem 2 Let (H5) hold. Assume $h \in C([0,1],[0,\infty))$ and it is not identically zero on any subinterval of $[0,1]$, $f \in C(\mathbb{R},\mathbb{R})$ with $sf(s)>0$ for $s\neq 0$. Then for each $k\in\mathbb{N}$ and each $\nu\in\{+,-\}$, the connected component D_k^{ν} of $\varphi_k^{\nu}\cup\left(\left[\frac{\gamma_k}{f^0},\ \frac{\gamma_k}{f_0}\right]\times\{0\}\right)$ containing $\left[\frac{\gamma_k}{f^0},\ \frac{\gamma_k}{f_0}\right]\times\{0\}$ is unbounded, and lies in $(\mathbb{R}\times S_k^{\nu})\cup\left(\left[\frac{\gamma_k}{f^0},\ \frac{\gamma_k}{f_0}\right]\times\{0\}\right)$, where $f^0=\limsup_{|s|\to 0}\frac{f(s)}{s}\in(0,\infty)$, $f_0=\liminf_{|s|\to 0}\frac{f(s)}{s}\in(0,\infty)$, and γ_k is the k-th eigenvalue of the linear problem

$$\begin{cases} Lu=\gamma h(t)u,\ t\in(0,1),\\ \text{(b.c.)}. \end{cases}$$

Proof Let $r_0=\frac{f^0+f_0}{2}$, $d_0=\frac{f^0-f_0}{2}$ and $\chi(s)=f(s)-r_0 s$, $s\in\mathbb{R}$ Then

$$\liminf_{|s|\to 0}\frac{\chi(s)}{s}=-d_0,\ \limsup_{|s|\to 0}\frac{\chi(s)}{s}=d_0,$$

and (5) can be rewritten as

$$\begin{cases} Lu=\gamma h(t)r_0u+\gamma h(t)\chi(u)+g(t,u,u',\gamma),\ t\in(0,1),\\ \text{(b.c.)}. \end{cases}$$

By considering the approximate problem

$$\begin{cases} Lu=\gamma h(t)r_0u+\gamma h(t)\chi(|u|^{\varepsilon}u)+g(t,u,u',\gamma),\ t\in(0,1),\\ \text{(b.c.)}, \end{cases}$$

and applying similar arguments[17,25], we can get the desired conclusions.

Remark 3 Under the same assumptions as Theorem 2, Ma *et al*[25] studied the global bifurcation phenomena of nonlinear Sturm-Liouville problem

$$\begin{cases} -(pu')'+qu=\gamma h(t)f(u),\ t\in(0,1),\\ \text{(b.c.)}, \end{cases}$$

and obtained the same results as Theorem 2. Our main findings Theorems 1 and 2 extend the results of Ref. [25] to Sturm-Liouville problems with more general nonlinearities.

2 Positive Solutions for Model (3)

In this section, Theorems 1 and 2 shall be applied to prove the existence of positive solutions of system (3), where f is not necessarily linearizable at 0. It is not difficult to see that

$$\begin{cases} -v''=\mu u,\ t\in(0,1),\\ v(0)=v(1)=0, \end{cases}$$

is equivalent to the integral equation

$$v(t)=\mu\int_0^1 K(t,s)u(s)\mathrm{d}s=:\mu Au(t),\ t\in[0,1], \tag{20}$$

where

$$K(t,s)=\begin{cases} t(1-s),\ 0\leqslant t\leqslant s\leqslant 1,\\ s(1-t),\ 0\leqslant s\leqslant t\leqslant 1. \end{cases}$$

Obviously, A: $C[0,1]\to C[0,1]$ is a linear compact operator. By (3) and (20) we have

$$\begin{cases} -u''+\lambda u=f(t,u)+\mu uAu,\ t\in(0,1),\\ u(0)=u(1)=0. \end{cases} \tag{21}$$

Now, if λ satisfies (H1), then (21) can be rewritten as

$$u(t)=\int_0^1 G(t,s)f(s,u(s))\mathrm{d}s+\mu\int_0^1\int_0^1 G(t,s)u(s)K(s,\tau)u(\tau)\mathrm{d}\tau\mathrm{d}s=:Tu(t),$$

where $G(t,s)$ is the Green's function[26] of

$$\begin{cases} -u''+\lambda u=0,\ t\in(0,1),\\ u(0)=u(1)=0. \end{cases}$$

Clearly, T:$C[0,1]\to C[0,1]$ is completely continuous.

Theorem 3 Let (H1)-(H4) hold. If

$$\gamma_1(\alpha)\alpha_0>\alpha_0+\beta^0, \tag{22}$$

then (3) admits at least one positive solution (u,v) for any $\mu\in(0,\infty)$. Here α, α_0, β^0 are defined as in Section 1 and $\gamma_1(\alpha)$ is the principal eigenvalue of

$$\begin{cases} -u''+\lambda u=\gamma\alpha(t)u,\ t\in(0,1),\\ u(0)=u(1)=0. \end{cases}$$

Proof Let us consider the parameterized problem

$$\begin{cases} -u''+\lambda u=\gamma f(t,u)+\gamma\cdot\mu uAu,\ t\in(0,1),\\ u(0)=u(1)=0, \end{cases} \tag{23}$$

where $\gamma\in\mathbb{R}$ is a parameter. It is clear that any solution of (23) of the form $(1,u)$ yields a solution u of (21). From (H2) it follows that $\mu uAu+f(t,u)=0$ for $u=0$ and $t\in[0,1]$, which implies $(\gamma,0)$ is always the solution of (23). Moreover, it is not difficult to verify that $g(t,u,\gamma)=\gamma\mu uAu$ satisfies (H5) since

$$\frac{\gamma\mu uAu}{\|u\|_1}\leqslant\frac{|\gamma|\cdot\mu\|A\|_0\|u\|_1^2}{\|u\|_1}=|\gamma|\cdot\mu\|A\|_0\|u\|_1\to 0,$$

as $\|u\|_1\to 0$, uniformly in $\gamma\in\Gamma$, for every bounded interval Γ, where $\|A\|_0$ denotes the norm of the operator A. Therefore, by Theorem 1 the connected component D_1^+ of $\varphi_1^+\cup(I_1\times\{0\})$ containing $I_1\times\{0\}$ is unbounded and lies in $(\mathbb{R}\times S_1^+)\cup(I_1\times\{0\})$. Moreover, $D_1^+\cap(\{0\}\times E)=\varnothing$ since $(0,0)$ is the only solution of (23) for $\gamma=0$.

Applying similar methods[16] with obvious changes, we conclude D_1^+ joins $I_1\times\{0\}$ to $(0,\infty)$. This together with (22) implies D_1^+ crosses the hyperplane $\{1\}\times E$ in $\mathbb{R}\times E$, and so (21) has at least one positive solution u for any

$\mu \in (0, \infty)$. Moreover, it follows from (20) that system (3) has at least one positive solution (u, v) for any $\mu \in (0, \infty)$.

Theorem 4 Let (H1) hold. Assume that $h \in C([0, 1], [0, \infty))$ and it is not identically zero on any subinterval of $[0, 1]$, $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$. If $f^0 < \gamma_1$, then

$$\begin{cases} -u'' = uv - \lambda u + h(t) f(u), & t \in (0, 1), \\ -v'' = \mu u, & t \in (0, 1), \\ u(0) = u(1) = 0, \\ v(0) = v(1) = 0, \end{cases} \tag{24}$$

admits at least one positive solution (u, v) for any $\mu \in (0, \infty)$, where f^0 and γ_1 are defined as in Theorem 2.

Proof Obviously, (24) can be rewritten as the equivalent problem

$$\begin{cases} -u'' + \lambda u = h(t) f(u) + \mu u A u, & t \in (0, 1), \\ u(0) = u(1) = 0. \end{cases} \tag{25}$$

Let us consider the parameterized problem

$$\begin{cases} -u'' + \lambda u = \gamma h(t) f(u) + \gamma \cdot \mu u A u, & t \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{26}$$

where $\gamma \in \mathbb{R}$ is a parameter. Clearly, any solution of (26) of the form $(1, u)$ yields a solution u of (25). In addition, $\mu u A u + h(t) f(u) = 0$ for $u = 0$ and $t \in [0, 1]$, which means $(\gamma, 0)$ is always the solution of (26). It is not difficult to check that the assumptions of Theorem 2 are all satisfied. Thus, by Theorem 2 the connected component D_1^+ of $\varphi_1^+ \cup \left(\left[\frac{\gamma_1}{f^0}, \frac{\gamma_1}{f_0} \right] \times \{0\} \right)$ containing $\left[\frac{\gamma_1}{f^0}, \frac{\gamma_1}{f_0} \right] \times \{0\}$, is unbounded and lies in $(\mathbb{R} \times S_1^+) \cup \left(\left[\frac{\gamma_1}{f^0}, \frac{\gamma_1}{f_0} \right] \times \{0\} \right)$. The rest of the proof is similar to that of Theorem 3.1 in Ref. [16].

Remark 4 We would like to point out that by applying Theorems 1 and 2, and using similar discussions as in the proofs of Theorems 3 and 4, we can also establish the existence of nodal solutions for systems (3) and (24).

Remark 5 In the special case $f^0 = f_0 = \delta \in (0, \infty)$, Chen and Ma[16] studied the existence of positive solutions of (24), where bifurcation occurs from a single point $\left(\frac{\gamma_1}{\delta}, 0 \right)$ and they showed (24) admits at least one positive solution for any $\mu \in (0, \infty)$, which has been extended by Theorem 4 to the case where f is not necessarily linearizable at 0.

3 Related Results

We investigate in this section the non-cooperative model

$$\begin{cases} -u'' = uv - \lambda u + f(t, u), & t \in (0, 1), \\ -v'' = \mu u - cv, & t \in (0, 1), \end{cases} \tag{27}$$

which is related to the stationary version of the reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = u(av - b), & x \in \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} - \Delta v = du - cv, & x \in \Omega, \ t > 0. \end{cases} \tag{28}$$

Clearly, (28) extends the model (1) by adding the diffusion and the linear feedback to the temperature. Suppose the boundary of the closed container is heat-insulated and neutron flux can not go through the boundary, then the proper boundary conditions are Neumann boundary conditions

$$\frac{\partial u}{\partial \vec{n}} = 0 = \frac{\partial v}{\partial \vec{n}}, \ x \in \partial\Omega, \ t > 0,$$

where $\vec{n}$ indicates the outward normal. Neumann boundary conditions have been given special attention as the behavior of the solution for this type of boundary conditions is quite different from those for other types of boundary conditions[2-4,6,7,16].

In the following, we establish the existence of positive solutions of (27) subject to the Neumann boundary conditions, namely,

$$\begin{cases} -u'' = uv - \lambda u + f(t, u), & t \in (0, 1), \\ -v'' = \mu u - cv, & t \in (0, 1), \\ u'(0) = 0 = u'(1), \\ v'(0) = 0 = v'(1). \end{cases} \tag{29}$$

Assume that

(H1)′ λ, $c \in \left(-\frac{\pi^2}{4}, 0 \right) \cup (0, \infty)$ are constants.

Then (29) can be transformed into

$$\begin{cases} -u'' + \lambda u = \mu u \tilde{L} u + f(t, u), & t \in (0, 1), \\ u'(0) = 0 = u'(1), \end{cases}$$

where $\tilde{L}: C[0, 1] \to C[0, 1]$ is a linear compact operator. We first apply Theorems 1 and 2. Following the argument used in the proofs of Theorems 3.1 and 3.2 in Ref. [16], with obvious changes, we obtain

Theorem 5 Let (H1)′, (H2), (H3) and (H4) hold. If $\eta_1(\alpha)\alpha_0 > \alpha_0 + \beta^0$, then (29) admits at least one positive solution (u, v) for any $\mu \in (0, \infty)$. Here α, α_0, β^0 are defined as in Section 1 and $\eta_1(\alpha)$ is the principal eigenvalue of

$$\begin{cases} -u'' + \lambda u = \eta \alpha(t) u, & t \in (0, 1), \\ u'(0) = 0 = u'(1). \end{cases}$$

Theorem 6 Let (H1)′ hold. Assume $h \in C([0, 1],$

$[0,\infty))$ and it is not identically zero on any subinterval of $[0, 1]$, $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$. If $f^0 < \tau_1$, then

$$\begin{cases} -u'' = uv - \lambda u + h(t) f(u), & t \in (0, 1), \\ -v'' = \mu u - cv, & t \in (0, 1), \\ u'(0) = 0 = u'(1), \\ v'(0) = 0 = v'(1), \end{cases}$$

has at least one positive solution (u, v) for any $\mu \in (0, \infty)$, where f^0 is defined as in Theorem 2 and τ_1 is the principal eigenvalue of

$$\begin{cases} -u'' + \lambda u = \tau h(t) u, & t \in (0, 1), \\ u'(0) = 0 = u'(1). \end{cases}$$

Finally, if we consider the nuclear reactors exchanging heat energy with the outside, then the proper boundary condition is

$$\frac{\partial u}{\partial \vec{n}} = 0, \quad \frac{\partial v}{\partial \vec{n}} + \rho v = 0, \quad x \in \partial\Omega, \quad t > 0,$$

where $\rho > 0$ is the heat transfer coefficient. Under the assumption of nonnegative initial values, several authors have studied the system

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = u(av - b), & x \in \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} - \Delta v = du - cv, & x \in \Omega, \ t > 0, \\ \dfrac{\partial u}{\partial \vec{n}} = 0, \ \dfrac{\partial v}{\partial \vec{n}} + \rho v = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x) \geqslant 0, \ v(x, 0) = v_0(x) \geqslant 0, & x \in \bar{\Omega}, \end{cases} \tag{30}$$

which is a mathematical model much closer to reality[3-4]. We are interested in the existence of positive solutions of model

$$\begin{cases} -u'' = uv - \lambda u + f(t, u), & t \in (0, 1), \\ -v'' = \mu u - cv, & t \in (0, 1), \\ u'(0) = 0 = u'(1), \\ v'(0) + \rho v(0) = 0, \ v'(1) + \rho v(1) = 0, \end{cases} \tag{31}$$

which is related to the stationary version of the reaction-diffusion system (30). Suppose

(H1)″ $c > 0$ is a constant and $0 < \rho < m := \sqrt{c}$.

Then it is not difficult to check that (31) can be rewritten as

$$\begin{cases} -u'' + \lambda u = \mu u L_0 u + f(t, u), & t \in (0, 1), \\ u'(0) = 0 = u'(1), \end{cases}$$

where $L_0 : C[0, 1] \to C[0, 1]$ is a linear compact positive operator. By an argument similar to that of Theorems 3 and 4, we can establish the following

Theorem 7 Let (H1)′, (H2), (H3) and (H4) hold. If $\eta_1(\alpha)\alpha_0 > \alpha_0 + \beta^0$, then (31) has at least one positive solution (u, v) for any $\mu \in (0, \infty)$. Here α, α_0, β^0 are defined as in Section 1 and $\eta_1(\alpha)$ is given by Theorem 5.

Theorem 8 Let (H1)″ hold. Assume that $h \in C([0, 1], [0, \infty))$ and it is not identically zero on any subinterval of $[0, 1]$, $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$. If $f^0 < \tau_1$, then

$$\begin{cases} -u'' = uv - \lambda u + h(t) f(u), & t \in (0, 1), \\ -v'' = \mu u - cv, & t \in (0, 1), \\ u'(0) = 0 = u'(1), \\ v'(0) + \rho v(0) = 0, \ v'(1) + \rho v(1) = 0, \end{cases}$$

has at least one positive solution (u, v) for any $\mu \in (0, \infty)$, where f^0 and τ_1 are defined as in Theorem 2 and Theorem 6, respectively.

Remark 6 The results of Theorems 3 and 4 remain true for more general models (27) and

$$\begin{cases} -u'' = uv - \lambda u + h(t) f(u), & t \in (0, 1), \\ -v'' = \mu u - cv, & t \in (0, 1), \end{cases}$$

subject to Dirichlet boundary conditions, where c is a constant satisfying (H1).

Remark 7 Let $\Omega = \{x \in \mathbb{R}^n : 0 < r < |x| < R\}$ be an annulus in $\mathbb{R}^n$ with $n \geqslant 2$, where r and R are constants denoting the inner and outer radii of the annulus, respectively. Then the results of the paper hold for corresponding high-dimensional systems.

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一个新的全局分歧定理及其在核反应堆模型中的应用

陈瑞鹏

北方民族大学 数学与信息科学学院，宁夏 银川 750021

摘要：本文针对非线性Sturm-Liouville问题 $\begin{cases} -(pu')'+qu=\gamma f(t,u)+g(t,u,u',\gamma),\ t\in(0,1) \\ b_0u(0)+c_0u'(0)=0,\ b_1u(1)+c_1u'(1)=0 \end{cases}$ 建立了全局分歧定理，其中f在0处不是Fréchet可微的。该定理表明分歧现象在某些区间$I_k(k\in\mathbb{N})$发生，且包含$I_k\times\{0\}$的无界连通分支D_k^{ν}位于$(\mathbb{R}\times S_k^{\nu})\cup(I_k\times\{0\})$之中。应用上述全局分歧定理，我们证明了对于任意参数$\mu\in(0,\infty)$，非合作核反应堆模型 $\begin{cases} -u''=uv-\lambda u+f(t,u),\ t\in(0,1) \\ -v''=\mu u,\qquad\qquad\quad t\in(0,1) \end{cases}$ 至少存在一个正解。

关键词：非合作模型；核反应堆；分歧；正解；存在性

□